Long-time behaviors of mean-field interacting particle systems and McKean-Vlasov equations

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Based on joint works with A. Guillin, L. Wu and C. Zhang:

- **Guillin-L.-Wu-Zhang, Ann. Appl. Prob., 2022**
  Uniform Poincaré inequalities and logarithmic Sobolev inequalities for mean field particle systems
- **L.-Wu-Zhang, Commun. Math. Phys., 2021**
  Long-time behaviors of mean-field interacting particle systems related to McKean-Vlasov equation
- **L.-Wu, Stoch. Proc. Appl., 2020**
  Large deviations for empirical measures of mean-field Gibbs measures
- **Guillin-L.-Wu-Zhang, J. Math. Purés Appl., 2021**
  The kinetic Fokker-Planck equation with mean field interaction

Outline

1. Introduction
2. Coupling method
   - Exponential convergence in W_1 distance
   - Examples
3. Functional inequalities
   - Uniform log-Sobolev inequality
   - Exponential convergence of McKean-Vlasov equation in entropy

Mean-field interacting particle system

Consider the following interacting particle system:

$$dX_t^{i,N} = b_t(X_t^{i,N}, L_t^N)dt + \sigma_t(X_t^{i,N}, L_t^N)dB_t^i, 1 \le i \le N$$

- $L_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ empirical measure
- coefficients

$$b : \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$$
$$\sigma : \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times m},$$

- $B^1, \cdots, B^N$ independent BMs in $\mathbb{R}^d$
- $X_0^{1,N}, \cdots, X_0^{N,N}$ i.i.d., and independent of the BMs

♠ Two fundamental problems

- $t \to \infty$: Long time behaviors (equilibrium)
- $N \to \infty$: Macroscopic limit (part of Hilbert's 6th problem)

From Microscopic to Macroscopic

♢ Macroscopic limit as $N \to \infty$ – part of Hilbert's 6th problem.

- Macroscopic limit: Hydrodynamic limit, Thermodynamic limit, Bose-Einstein condensation
- Boltzmann, Landau, McKean-Vlasov, Vlasov-type Kinetic equations, etc.
- Convergence rate
- Curse of dimensionality

♡ Mean field limit - McKean-Vlasov SDE

$$dX_t = b_t(X_t, \mu_t)dt + \sigma_t(X_t, \mu_t)dB_t$$

where μ_t is the law of X_t.

Mean field limit

Mean field interacting particle system:

$$dX_t^{i,N} = b_t(X_t^{i,N}, L_t^N)dt + \sigma_t(X_t^{i,N}, L_t^N)dB_t^i, 1 \leq i \leq N \quad (1)$$

Self-interacting nonlinear diffusion or distribution dependent SDE:

$$dX_t = b_t(X_t, \mu_t)dt + \sigma_t(X_t, \mu_t)dB_t \quad (2)$$

Kac's Propagation of chaos (as $N \to +\infty$)

- $X_t^{i,N} \Longrightarrow \mu_t$ in law
- $L_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}} \Longrightarrow \mu_t$ in law
- Particles get independent when $N \to +\infty$ for k fixed:

$$(X_t^{1,N}, \cdots, X_t^{k,N}) \Longrightarrow (\mu_t)^{\otimes k} \text{ in law}$$

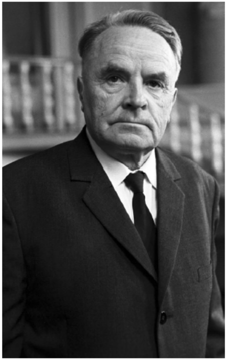
Applications

Mean-field interacting particle systems have been extensively studied in recent 40 years due to their wide range of applications in several fields including *physics, chemistry, biology, economics, mean-field games, financial mathematics, social science, machine learning* and so on.

- Physics, Chemistry: ions and electrons in plasmas, molecules in a fluid, galaxies in large scale cosmological models
- Biology: collective behaviors, neuronal network
- Economics, finances and Social Science: opinion dynamics, consensus model, mean field games
- Machine learning: deep learning, artificial neural network, distribution sampling algorithm, stochastic algorithm
- etc...

History

(a) Mark Kac

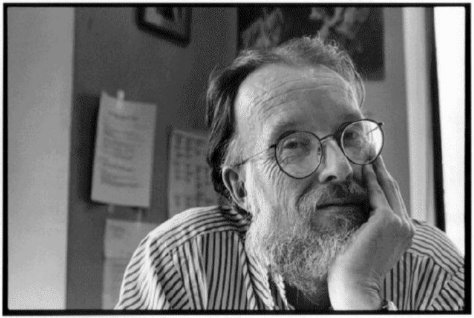


(b) Anatoly Vlasov

The story of these processes started with a stochastic toy model for the Vlasov equation of plasma proposed by Mark Kac in his paper *"Foundations of kinetic theory (1956)"*.

Figure:

History



In 1966 Henry P. McKean published his seminal paper *"A class of Markov processes associated with non-linear parabolic equations"*.

Figure:

Known results

- Kac (1956,1958), stochastic toy model for the Vlasov kinetic equation of plasma
- McKean (1966,1967), non-linear parabolic equations
- Sznitman (1991), Topics in propagation of chaos, Saint-Flour lecture notes
- Propagation of chaos: Macroscopic limit of the interacting particle system as $N \to \infty$. Méléard (1996), Benachour et al. (1998), Malrieu (2001,2003), Bolley et al. (2007,2010), Cattiaux et al. (2008), Jabin-Wang (2018), Durmus et al. (2020), L.-Wu-Zhang (2021), Lacker (2021), Delarue-Tse (2021), Guillin et al. (2021), etc.
- Existence and uniqueness, well-posedness, smoothness and regularization of the solutions, F.Y. Wang (2018-), X. Zhang and Röckner (2021), Li-Li-Xie (2020), Hammersley et al. (2021), Mishura-Veretennikov (2018), Buckdahn-Li-Peng-Rainer (2017), etc. (and the references therein)

Continued

- Large and moderate deviation principles:
  - **Empirical measure**: Léonard (1987) , S. Feng (1994), Dupuis et al. (2015), J. Reygner (2018), L.-Wu (2020)
  - **Weak interacting diffusions**: Dawson-Gärtner (1987), Budhiraja-Dupuis-Fischer (2012), Hoeksema et al. (2020), L.-Wu (2023+),
  - **Freidlin-Wentzell type LDP for McKean-Vlasov SDEs**: Herrmann et al. (2008), Dos Reis et al. (2019), Yuan-suo (2019), L.-Song-Zhai-Zhang (2022), L.-Qiao-Zhu, Cheng-L.-Zhu (2023) etc.
- **CLT** Wang-Zhao-Zhu (2021), Yuan-Suo (2021)

Continued

- Long time behaviors of the McKean-Vlasov SDE and Interacting particle system as $t \to \infty$. Carrillo-McCann-Villani (2003), Eberle et al. (2016, 2018), Luo-Wang (2016), Liang-Majka-Wang (2019), Liu-Wu-Zhang (2021), Guilin-L.-Wu-Zhang (2022), etc.
- Functional inequalities: Malrieu (2001,2003), Huang-Wang (2021), Guilin-L.-Wu-Zhang (2022), F.Y. Wang (2023).
- Kinetic case: Villani(2009), Mouhot et al. (2015, 2016), Herzog-Mattingly (2019), Iu-Mattingly (2020), Guillin-L.-Wu-Zhang (2021), Bao-Wang (2023)
- Others: numerical approximation, slow-fast, switching regime, path-dependent, delayed, with reflection, SPDE, Mean-field control...
- Other noises: fractional BM, Lévy, α-stable, GBM...

Our model: McKean-Vlasov equation with additive noise

- Consider the following nonlinear McKean-Vlasov equation

$$dX_t = \sqrt{2}dB_t - \nabla V(X_t)dt - \nabla_x W \circledast \mu_t(X_t)dt \qquad (3)$$

  - confinement potential $V : \mathbb{R}^d \to \mathbb{R}$
  - interaction potential $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $W(x, y) = W(y, x)$
  - $$\nabla_x W \circledast \mu_t(X_t) := \int_{\mathbb{R}^d} \nabla_x W(X_t, y)\mu_t(dy)$$

- The corresponding nonlinear Fokker-Planck equation

$$\partial_t \mu_t = \Delta \mu_t + \nabla \cdot [\mu_t(\nabla V + \nabla_x W \circledast \mu_t)] \qquad (4)$$

Long time behavior

Two important questions:

- Existence and uniqueness of the equilibrium state, i.e. the limit $\mu_\infty := \lim_{t\to\infty} \mu_t$. μ_∞ satisfies the following stationary equation (see Liu-Wu 2020 SPA):

$$\triangle \mu_\infty + \nabla \cdot [\mu_t(\nabla V + \nabla_x W \circledast \mu_t)] = 0$$

$$\mu_\infty(dx) = \exp(-V(x) - W * \mu_\infty(x))dx/C.$$

- Convergence rate to the limit (exponential or algebraic).

Free energy

- The free energy of the state ν is given by

$$\begin{aligned} E_W(\nu) &:= H(\nu|\alpha) + \frac{1}{2}\iint W(x,y)d\nu(x)d\nu(y) \\ &= H(\nu|\exp(-V(x) - \frac{1}{2}W \circledast \nu)) + c \end{aligned} \tag{5}$$

where

$$\alpha(dx) = e^{-V(x)}dx/C.$$

- The solution μ_∞ of the stationary equation of the nonlinear McKean-Vlasov equation is the **critical point of E_W**.
- Mean field entropy

$$H_W(\nu) := E_W(\nu) - \inf_{\mu \in \mathcal{M}_1(S)} E_W(\mu), \ \nu \in \mathcal{M}_1(S) \tag{6}$$

is just the **LDP rate function** of the empirical measure for mean field Gibbs measure (L.-Wu 2020SPA).

Carrillo-McCann-Villani's result 2003RMI

Assume that

$$\nabla^2 V \geq \gamma I, \gamma > 0$$

and $W(x, y) = W_0(x - y)$ with W_0 even and convex. Then

$$H_w(\mu_t) \leq e^{-\gamma t} H_W(\mu_0).$$

Strategy of the proofs:

- Gradient flow of the free energy, i.e.

$$\partial_t \mu_t = -\nabla H_W(\mu_t).$$

- Strictly displacement convex along geodesic, i.e.

$$\nabla^2 H_W(\mu_t) \geq C > 0.$$

- Logarithmic Sobolev inequalities and mass transportation inequalities, via either the *Bakry-Emery method*.

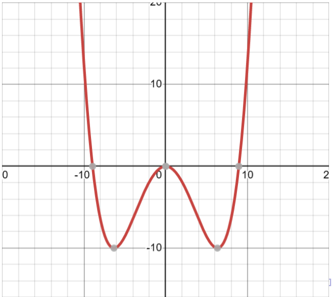
Our motivations and goals

Remove the convexity assumption on V and W!

Example: Curie-Weiss model

$$V(x) = \beta(x^4/4 - x^2/2), \ W(x,y) = -\beta Kxy$$

where $\beta > 0$ is the inverse temperature.
This model is called ferromagnetic ($K > 0$) or anti-ferromagnetic ($K < 0$).



Mean field model

McKean-Vlasov SDE:

$$\begin{cases} dX_t = \sqrt{2}dB_t - \nabla V(X_t)dt - \nabla_x W \circledast \mu_t(X_t)dt, \\ \nabla_x W \circledast \mu_t(X_t) = \int \nabla_x W(X_t, y)\mu_t(dy), \end{cases}$$

Mean field interacting particle system

$$\begin{cases} dX_t^{i,N} = \sqrt{2}dB_t^i - \nabla V(X_t^{i,N})dt - \frac{1}{N-1}\sum_{j:j\neq i, 1\leq j\leq N} \nabla_x W(X_t^{i,N}, X_t^{j,N})dt, \\ X_0^{i,N} = X_0^i, \; i = 1, \cdots, N, \end{cases}$$

Our Strategy:

- Long-time behavior of IPS: Coupling method, Functional inequalities.
- From IPS to McKean-Vlasov: Propagation of chaos.

Propagation of chaos - classic results

- Mean field interacting particle system

$$dX_t^{i,N} = b(X_t^{i,N}, L_t^N)dt + \sigma(X_t^{i,N}, L_t^N)dB_t^i \qquad (7)$$

- Consider the following independent particle system:

$$d\bar{X}_t^i = b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)dB_t^i, 1 \le i \le N \qquad (8)$$

  - $\bar{X}_0^1, \cdots, \bar{X}_0^N$ i.i.d., and independent of the BMs
  - $\bar{X}_t^1, \cdots, \bar{X}_t^N$ i.i.d. with common law μ_t
- Sznitman (1991, Synchronous Coupling) when b is bounded Lipschitz and σ is constant

$$\mathbb{E}[\sup_{0 \le t \le T} |X_t^{i,N} - \bar{X}_t^i|] \le \frac{C(T)}{\sqrt{N}}.$$

Other results by Méléard (1996), Benachour et al. (1998)... in bounded time intervals, but Not uniform in time!

Uniform in time propagation of chaos

Uniform in time propagation of chaos is much more difficult!

- For any $t \geq 0$ and $1 \leq i \leq N$,

$$W_p(\mu_t^{i,N}, \mu_t) \leq \frac{C}{\sqrt{N}}, p = 1, 2$$

-

$$H(\mu_t^{i,N} | \mu_t) \leq \frac{C}{\sqrt{N}}.$$

- Convex potentials: Malrieu (2001, 2003), Cattiaux et al. (2008), Bolley et al. (2010), Lacker (2023)
- Non-convex potentials: Durmus et al. (2020), L.-Wu-Zhang (2021 CMP)
- Singular case: Jabin-Wang (2018), Delarue-Tse (2021),Guillin et al. (2021), Hao-Röckner-Zhang (2022)
- Methods: Coupling, Functional inequalities, BBGKY Strategy...

Coupling method

Approximate componentwise reflection coupling

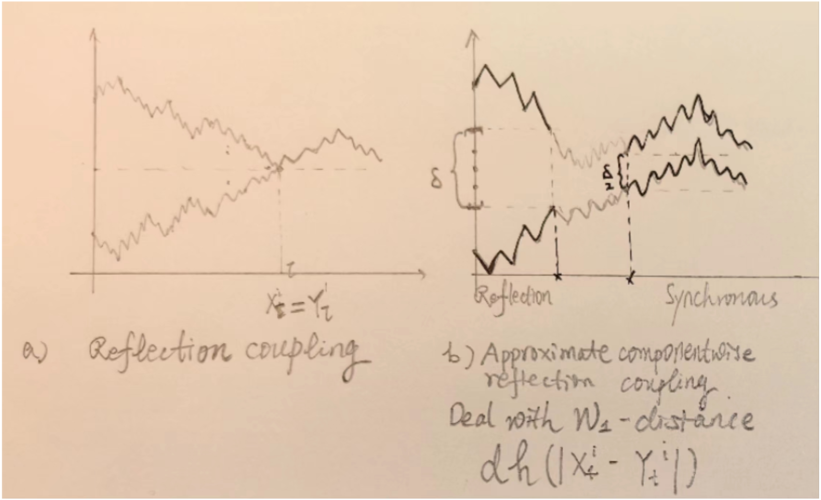


Figure:

Approximate componentwise reflection coupling

First introduced by A.Eberle (PTRF 2016)

$$
\begin{aligned}
dX_t^{i,N} &= \sqrt{2}[\lambda_\delta(|Z_t^i|)dB_t^{1,i} + \pi_\delta(|Z_t^i|)dB_t^{2,i}] - \nabla V(X_t^{i,N})dt \\
&\quad - \frac{1}{N-1}\sum_{j:j\neq i,1\leq j\leq N} \nabla_x W(X_t^{i,N}, X_t^{j,N})dt, \\
dY_t^{i,N} &= \sqrt{2}[\lambda_\delta(|Z_t^i|)R_t^i dB_t^{1,i} + \pi_\delta(|Z_t^i|)dB_t^{2,i}] - \nabla V(Y_t^{i,N})dt \\
&\quad - \frac{1}{N-1}\sum_{j:j\neq i,1\leq j\leq N} \nabla_x W(Y_t^{i,N}, Y_t^{j,N})dt,
\end{aligned} \tag{9}
$$

- $\lambda_\delta(r)^2 + \pi_\delta(r)^2 = 1$, $\lambda_\delta(r) = 1$ if $r \geq \delta$, $\lambda_\delta(r) = 0$ if $r \leq \delta/2$.
- $Z_t^i := X_t^{i,N} - Y_t^{i,N}$ and $R_t^i := I_d - 2e_t^i(e_t^i)^T$, where $e_t^i(e_t^i)^T$ is the orthogonal projection onto the unit vector $e_t^i := Z_t^i/|Z_t^i|$.
- Strategy for proofs: choose appropriate reference h and use Itô's formula for $h(|Z_t^i|)$. Then let $\delta \to 0$.

Notations of IPS

- $\mathbb{P}_x$: law of $X^{(N)} = (X_t^{1,N}, \cdots, X_t^{N,N})_{t\geq 0}$ with initial value $X_0^{(N)} = x \in (\mathbb{R}^d)^N$.
- $\{P_t^{(N)}\}_{t\geq 0}$: transition semigroup
- $\mathcal{L}^{(N)}$: generator of $(X_t^{1,N}, \cdots, X_t^{N,N})$ is given by

$$\begin{aligned}\mathcal{L}^{(N)} f(x_1, \cdots, x_N) &= \sum_{i=1}^N \mathcal{L}_i^{(N)} f(x_1, \cdots, x_N) \\ \mathcal{L}_i^{(N)} f(x_1, \cdots, x_N) &:= \Delta_i f(x_1, \cdots, x_N) - \nabla_i V(x_i) \cdot \nabla_i f(x_1, \cdots, x_N) \\ &\quad - \frac{1}{N-1} \sum_{j\neq i} (\nabla_x W)(x_i, x_j) \cdot \nabla_i f(x_1, \cdots, x_N)\end{aligned} \tag{10}$$

for any smooth function f on $(\mathbb{R}^d)^N$.

Invariant probability measure

- The unique invariant probability measure:

$$\mu^{(N)}(dx_1, \cdots, dx_N) = \frac{1}{Z_N} \exp\{-H_N(x_1, \cdots, x_N)\} dx_1 \cdots dx_N$$

where

$$H_N(x_1, \cdots, x_N) := \sum_{i=1}^{N} V(x_i) + \frac{1}{N-1} \sum_{1 \le i < j \le N} W(x_i, x_j)$$

is the Hamiltonian, and Z_N is the normalization constant called *partition function* in statistical mechanics, which is assumed to be finite.

- Without interaction (i.e. $W = 0$ or constant), $\mu^{(N)} = \alpha^{\otimes N}$ (i.e. the particles are independent), where

$$d\alpha(x) = \frac{1}{C} e^{-V(x)} dx, \ \ C = \int_{\mathbb{R}^d} e^{-V(x)} dx.$$

Conditions on the dissipative rate

- **Dissipative rate** $b_0(r)$ of the drift of one single particle at distance $r > 0$,

$$\langle x - y, -[\nabla V(x) - \nabla V(y)] - [\nabla_x W(x,z) - \nabla_x W(y,z)]\rangle \leq b_0(r)|x-y| \tag{11}$$

  holds for any $x, y, z \in \mathbb{R}^d$ with $|x - y| = r$.

- Assume that $b_0(r)$ is a continuous function on $(0, +\infty)$ satisfying

$$\limsup_{r\to+\infty} \frac{b_0(r)}{r} < 0, \tag{12}$$

  i.e. the drift of one particle is *dissipative at infinity*.

Important reference function h for coupling method

First introduced by Wu. 2009 JFA

- Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be the function determined by: $h(0) = 0$ and

$$h'(r) = \frac{1}{4} \exp\left(-\frac{1}{4}\int_0^r b_0(s)ds\right) \int_r^{+\infty} s\cdot\exp\left(\frac{1}{4}\int_0^s b_0(u)du\right) ds. \tag{13}$$

It is a well defined C^2 function by the dissipative condition (12).

- For any function $f \in C^2(0,+\infty)$ and $r > 0$, let $\mathcal{L}_{ref}$ be the generator defined by

$$\mathcal{L}_{ref} f(r) := 4f''(r) + b_0(r)f'(r). \tag{14}$$

- h is a solution of the *one-dimensional Poisson equation*

$$\mathcal{L}_{ref} h(r) = 4h''(r) + b_0(r)h'(r) = -r, \ \ r > 0 \tag{15}$$

with $h(0) = 0$.

Key assumption

We make the following key assumption on the interaction potential:

$$\textbf{(H)}: \ \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty < 1$$

where $\nabla^2_{xy} W = (\frac{\partial^2}{\partial x_i \partial y_j} W)_{1 \le i,j \le d}$, and

$$\|\nabla^2_{xy} W\|_\infty := \sup_{x,y \in \mathbb{R}^d} \sup_{z \in \mathbb{R}^d, |z|=1} |\nabla^2_{xy} W(x,y) z|.$$

- When the dissipativity at infinity condition (12) is satisfied, $b_0(r)$ can be taken as $-c_1 r + c_2$ (with $c_1, c_2 > 0$), so $\|h'\|_\infty := \sup_{r \ge 0} h'(r) < +\infty$.
- Notice that under the assumption **(H)** and dissipative condition (12), both the McKean-Vlasov SDE and the mean field interacting particle system have *unique strong solutions*.

Key assumption - continued

We make the following key assumption on the interaction potential:

$$\textbf{(H)}: \ \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty < 1$$

where $\nabla^2_{xy} W = (\frac{\partial^2}{\partial x_i \partial y_j} W)_{1\le i,j\le d}$, and

$$\|\nabla^2_{xy} W\|_\infty := \sup_{x,y\in\mathbb{R}^d} \sup_{z\in\mathbb{R}^d, |z|=1} |\nabla^2_{xy} W(x,y)z|.$$

- Condition **(H)** is a translation of Dobrushin-Zegarlinski's uniqueness condition in the framework of mean field, and it implies that the mean field has *no phase of transition* (see Guillin-L.-Wu-Zhang(2022 AAP)).
- This condition generalizes the convexity requirement of V and W in the previous results, which is our main contribution.

Exponential convergence

Theorem

Assume (12) and (**H**)*. Suppose that* $\exists M \in \mathbb{R}$ *s.t.*

$$b_0(r) \leq rM, \forall r > 0 \tag{16}$$

then for any $\varepsilon > 0$ *such that*

$$K_\varepsilon := \frac{1 - \|\nabla^2_{xy} W\|_\infty \|h'\|_\infty - \varepsilon(M + \|\nabla^2_{xy} W\|_\infty)}{\|h'\|_\infty + \varepsilon} > 0, \tag{17}$$

we have for any $x_0, y_0 \in (\mathbb{R}^d)^N$

$$W_{d_{l^1}}(P_t^{(N)}(x_0,\cdot), P_t^{(N)}(y_0,\cdot)) \leq A_\varepsilon e^{-K_\varepsilon t} d_{l^1}(x_0, y_0), \ \forall t \geq 0, \tag{18}$$

where

$$A_\varepsilon = \sup_{r>0} \frac{r}{h(r) + \varepsilon r} \cdot \sup_{r>0} \frac{h(r) + \varepsilon r}{r}. \tag{19}$$

Uniform in time propagation of chaos

Theorem

Suppose that $b_0(r) \leq rM, \forall r > 0$, *for some* $M \in \mathbb{R}$. *For any* $\epsilon > 0$ *such that* $K_\epsilon > 0$, *and* $\tilde{\epsilon} \in (0, c_1 - c_3 - \|\nabla^2_{xy} W\|_\infty)$, *we have*

- **(Uniform in time propagation of chaos)** *for all time* $t > 0$ *and any* $1 \leq k \leq N$*:*

$$W_{1,d_{l^1}}(\mu_t^{\otimes k}, \mu_t^{[1,k],N}) \leq \frac{k}{\sqrt{N-1}} \frac{A_\epsilon}{K_\epsilon} \|\nabla^2_{xy} W\|_\infty (m_2(\mu_0) \wedge \hat{c}(\epsilon)) \quad (20)$$

where $\mu_t = u_t dx$ *is the solution of the McKean-Vlasov equation, and* $\mu_t^{[1,k],N}$ *is the joint law of the* k *particles* $(X_t^{i,N}, 1 \leq i \leq k)$ *in the mean-field system of interacting particles* $(X_t^{i,N})_{1 \leq i \leq N}$ *with* $X_0^{i,N}, 1 \leq i \leq N$ *i.i.d. of law* μ_0 *(independent of* $(B_t^{i,N})_{1 \leq i \leq N, t \geq 0}$*).*

Path-type propagation of chaos

Theorem

- **(Path-type propagation of chaos)** *for any* $T > 0$, $1 \le k \le N$, *denote* $P_\nu(\cdot) = \int_{(\mathbb{R}^d)^N} P_x(\cdot) d\nu(x)$ *the law of* $(X_t^{(N)})_{t \ge 0}$ *with the initial distribution* ν, $P_\nu^{[1,k],N}|_{[0,T]}$ *the joint law of paths of the* k *particles* $((X_t^{i,N})_{t \in [0,T]}, 1 \le i \le k)$ *in time interval* $[0, T]$, *and* Q_{μ_0} *the law of the self-interacting diffusion* $(X_t)_{t \ge 0}$ *with the initial distribution* μ_0. *We have*

$$W_{1,d_{L^1[0,T]}}\left(P_{\mu_0^{\otimes N}}^{[1,k],N}|_{[0,T]}, Q_{\mu_0}^{\otimes k}|_{[0,T]}\right) \le \frac{kT}{\sqrt{N-1}} \frac{\|\nabla_{xy}^2 W\|_\infty \|h'\|_\infty}{1 - \|\nabla_{xy}^2 W\|_\infty \|h'\|_\infty} \cdot (m_2(\mu_0) \wedge \hat{c}(\epsilon)). \quad (21)$$

Exponential convergence of the nonlinear McKean-Vlasov equation

Corollary

Under the same assumptions as in the Theorem above, for any $\varepsilon > 0$ so that $K_\varepsilon > 0$, we have for the solutions μ_t, ν_t of the self-interacting diffusion (3) with the initial distributions μ_0, ν_0 which have finite second moments respectively,

$$W_1(\mu_t, \nu_t) \le A_\varepsilon e^{-K_\varepsilon t} W_1(\mu_0, \nu_0), \ \forall t \ge 0, \tag{22}$$

where K_ε and A_ε are given by (17) and (19) respectively. Especially, by taking $\nu_0 = \mu_\infty$, we get

$$W_1(\mu_t, \mu_\infty) \le A_\varepsilon e^{-K_\varepsilon t} W_1(\mu_0, \mu_\infty), \ \forall t \ge 0. \tag{23}$$

Example 1 - Curie-Weiss model

Let $d = 1$, and

$$V(x) = \beta(x^4/4 - x^2/2),\ \ W(x,y) = -\beta Kxy$$

where $\beta > 0$ is the inverse temperature. This model is ferromagnetic or anti-ferromagnetic according to $K > 0$ or $K < 0$.

- $b_0(r) = \beta r(1 - r^2/4)$, $\forall r > 0$. It is obvious that conditions on b_0 are satisfied and $b_0(r) \leq \beta r$ (i.e. $M = \beta$).
- $\|\nabla^2_{xy} W\|_\infty = |K|\beta$, assumption **(H)** holds once if

$$|K|\sqrt{\pi\beta}e^{\beta/4} \leq 1 \tag{24}$$

Example 2 - Double-Well confinement potential and quadratic interaction

Let $d = 1$, and

$$V(x) = \beta(x^4/4 - x^2/2), \ W(x,y) = \beta K(x-y)^2$$

where $\beta > 0$ is the inverse temperature, $K \in \mathbb{R}$.

- $b_0(r) = \beta r(1 - 2K - r^2/4)$, $\forall r > 0$. $M = \beta(1 - 2K)$
- Assumption **(H)** holds once if

$$\begin{cases} 2|K|\sqrt{\pi\beta}e^{(1-2K)^2\beta/4} \leq 1, & \text{if } \ K \leq \frac{1}{2} \\ 2|K|\sqrt{\pi\beta} \leq 1, & \text{if } \ K > \frac{1}{2}. \end{cases} \tag{25}$$

Functional inequalities

Functional inequalities

Functional inequalities such as *Poincaré*, *optimal transportation* or *logarithmic Sobolev inequalities* have nowadays an important impact on various fields of mathematics (probability, PDE, statistics,...) due to their various properties such as

- convergence to equilibrium (in L^2 or in entropy)
- concentration of measure (exponential or gaussian), see the book of Ledoux or F.Y. Wang.

Main goals: for the invariant measure $\mu^{(N)}$, we establish

♠ uniform Poincaré inequalities (in the number of particles N).

♡ uniform logarithmic Sobolev inequalities.

♣ exponential entropic decay for non-linear McKean-Vlasov equation, based on the uniform logarithmic Sobolev inequalities and propagation of chaos.

Relative entropy and Logarithmic Sobolev inequality

- The relative entropy of a probability measure ν w.r.t. the given probability measure μ on $\mathbb{R}^d$:

$$H(\nu|\mu) := \begin{cases} \int f^2 \log f^2 d\mu = \mathrm{Ent}_\mu(f^2), & \text{if } \nu \ll \mu, f^2 := \frac{d\nu}{d\mu} \\ +\infty, & \text{otherwise.} \end{cases} \quad (26)$$

- Logarithmic Sobolev inequality:

$$(LSI) \quad CEnt_\mu(f^2) \le 2 \int |\nabla f|^2 d\mu \quad (27)$$

- Exponential convergence of the Markov semigroups P_t

$$H(\mu_t|\mu) \le e^{-Ct} H(\mu_0|\mu).$$

PI and LSI

- One crucial property: *tensorization* (or dimension free), i.e.

$$\mu \text{ satisfies PI or LSI} \Rightarrow \mu^{\otimes N} \text{ satisfies the same inequality}$$

  with the same constant (and thus independent of N).
  This leads for example to
  - (non asymptotic) Gaussian deviation inequalities
  - convergence to equilibrium independent of the number of particles.
- However interesting physical systems are far from being independent, such as
  - spin systems
  - mean field models

  with a particular emphasis on the dependence on the number of spins or particles.

Framework and main assumptions

We work in the following framework.

(H1) The confinement potential $V : \mathbb{R}^d \to \mathbb{R}$ is C^2-smooth, $\mathrm{Hess}(V)$ is bounded from below and there are two positive constants c_1, c_2 such that

$$x \cdot \nabla V(x) \geq c_1 |x|^2 - c_2, \; x \in \mathbb{R}^d. \tag{28}$$

(H2) The pairwise interaction potential $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is C^2-smooth such that its Hessian $\nabla^2 W$ is bounded and

$$\iint_{(\mathbb{R}^d)^2} \exp\left(-[V(x) + V(y) + \lambda W(x,y)]\right) dxdy < +\infty, \; \forall \lambda > 0.$$

Framework and main assumptions - continued

(H3) **(Lipschitz spectral gap condition for one particle)** the following constant is finite

$$c_{Lip} := \frac{1}{4}\int_0^{\infty} \exp\left\{\frac{1}{4}\int_0^s b_0(u)\mathrm{d}u\right\} s\mathrm{d}s (= h'(0)) < +\infty \tag{29}$$

where $b_0(r)$ is the *dissipativity rate* of the drift of one particle in the system at distance $r > 0$:

$$b_0(r) = \sup_{x,y,z\in\mathbb{R}^d : |x-y|=r} -\langle \frac{x-y}{|x-y|}, (\nabla V(x) - \nabla V(y)) + (\nabla_x W(x,z) - \nabla_x W(y,z))\rangle. \tag{30}$$

Uniform log-sobolev inequality for mean-field $\mu^{(N)}$

Theorem

Assume that

- *for some best constant* $\rho_{\mathrm{LS}} > 0$*, the conditional marginal distributions* $\mu_i := \mu_i(dx_i|x^{\hat{i}})$ *on* $\mathbb{R}^d$ *satisfy the log-Sobolev inequality for all* i *and* $x^{\hat{i}}$ *;:*

$$\rho_{\mathrm{LS}} Ent_{\mu_i}(f^2) \leq 2 \int |\nabla f|^2 \mathrm{d}\mu_i \tag{31}$$

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$$\gamma_0 = c_{Lip} \sup_{x,y \in \mathbb{R}^d, |z|=1} |\nabla^2_{x,y} W(x,y) z| < 1. \tag{32}$$

then $\mu^{(N)}$ *satisfies*

$$\rho_{\mathrm{LS}}(1-\gamma_0)^2 Ent_{\mu^{(N)}}(f^2) \leq 2 \int_{(\mathbb{R}^d)^N} |\nabla f|^2 d\mu^{(N)} \tag{33}$$

Example: Curie-Weiss model

- $V(x) = \beta(x^4/4 - x^2/2)$, $W(x,y) = -\beta Kxy$ where $\beta > 0$.
- As given before we have

$$c_{Lip} \le \sqrt{\frac{\pi}{\beta}} e^{\beta/4}.$$

- So

$$\gamma_0 \le c_{Lip} \|\nabla^2_{x,y} W\|_\infty \le \sqrt{\pi\beta} e^{\beta/4} |K| < 1,$$

if β or K is sufficiently small.

log-Sobolev inequality: from IPS to McKean-Vlasov

- The log-Sobolev inequality for $\mu^{(N)}$ can be rewritten as

$$\rho_{LS}(\mu^{(N)})H(\nu|\mu^{(N)}) \leq 2I(\nu|\mu^{(N)}), \ \forall \nu \in \mathcal{M}_1((\mathbb{R}^d)^N). \quad (34)$$

From IPS to McKean-Vlasov: let $N \to \infty$!

- The *Fisher-Donsker-Varadhan's information* of ν w.r.t. μ is defined by

$$I(\nu|\mu) := \begin{cases} \int |\nabla \sqrt{f}|^2 d\mu, & \text{if } \nu \ll \mu, \sqrt{f} := \sqrt{\frac{d\nu}{d\mu}} \in H^1_\mu \\ +\infty, & \text{otherwise} \end{cases} \quad (35)$$

where

$$H^1_\mu := \{g \in L^2(\mu) : \int |\nabla g|^2 d\mu < +\infty\}$$

is the domain of the Dirichlet form $\mathbb{E}_\mu[g] = \int |\nabla g|^2 d\mu$.

Identification of the free energy as rate function

Lemma (L.-Wu 2020SPA)

For any probability measure ν on $\mathbb{R}^d$ such that $H(\nu|\alpha) < +\infty$,

$$\frac{1}{N} H(\nu^{\otimes N}|\mu^{(N)}) \to H_W(\nu), \text{ as } N \to +\infty. \tag{36}$$

Recall that

- $$E_f(\nu) := H(\nu|\alpha) + \frac{1}{2}\iint W(x,y)d\nu(x)d\nu(y)$$
- $$H_W(\nu) := E_W(\nu) - \inf_{\tilde{\nu}\in\mathcal{M}_1(\mathbb{R}^d)} E_W(\tilde{\nu})$$
- $H_W(\nu)$ can be identified as the mean relative entropy per particle of $\nu^{\otimes N}$ w.r.t. the mean field Gibbs measure $\mu^{(N)}$.

Fisher-Donsker-Varadhan's information

Lemma

(convergence of the Fisher information) *If* $I(\nu|\alpha) < +\infty$*, then*

$$\frac{1}{N} I(\nu^{\otimes N}|\mu^{(N)}) \to I_W(\nu), \text{ as } N \to +\infty. \quad (37)$$

$I_W(\nu)$ can be also interpreted as the mean Fisher-Donsker-Varadhan's information per particle:
If $d\nu(x) = f(x)dx$, $\int_{\mathbb{R}^d} |x|^2 d\nu(x) < +\infty$ and $\nabla f \in L^1_{loc}(\mathbb{R}^d)$ in the distribution sense,

$$I_W(\nu) := \frac{1}{4} \int_{\mathbb{R}^d} \left| \frac{\nabla f(x)}{f(x)} + \nabla V(x) + (\nabla_x W \circledast \nu)(x) \right|^2 d\nu(x), \quad (38)$$

and $+\infty$ otherwise.

LSI: from particles system to McKean-Vlasov

- $$\rho_{LS} H(\nu|\mu^{(N)}) \leq 2I(\nu|\mu^{(N)})$$
- $$\frac{1}{N} H(\nu^{\otimes N}|\mu^{(N)}) \to H_W(\nu), \text{ as } N \to +\infty.$$
- $$\frac{1}{N} I(\nu^{\otimes N}|\mu^{(N)}) \to I_W(\nu), \text{ as } N \to +\infty.$$

$\Rightarrow$ LSI for nonlinear McKean-Vlasov equation:

$$\rho_{LS} H_W(\nu) \leq 2I_W(\nu).$$

Exponential convergence of McKean-Vlasov equation

Theorem

Assume the uniform marginal log-Sobolev inequality, i.e. (31) with $\rho_{LS,m} > 0$, and the uniqueness condition (32). Then

(1) *there exists a unique minimizer μ_∞ of H_W over $\mathcal{M}_1(\mathbb{R}^d)$;*

(2) *the following (nonlinear) log-Sobolev inequality*

$$\rho H_W(\mu) \le 2I_W(\mu), \ \ \mu \in \mathcal{M}_1(\mathbb{R}^d) \tag{39}$$

holds, where

$$\rho := \limsup_{N\to\infty} \rho_{LS}(\mu^{(N)}) \ge \rho_{LS}(1-\gamma_0)^2;$$

(3) *given the initial distribution μ_0 of finite second moment,*

$$H_W(\mu_t) \le e^{-t\cdot\rho/2} H_W(\mu_0), \ \ t \ge 0 \tag{40}$$

Conclusions and future works

- Generalize the result of Carrillo-McCann-Villani(2003) from the *convex framework* $\longrightarrow$ *non-convex case*, by
  (1) first for the interacting particle system:
  - coupling method
  - functional inequalities

  (2) then nonlinear McKean-Vlasov equation by letting $N \to \infty$:
  - propagation of chaos.
- **Example**:

$$V(x) = \beta(x^4/4 - x^2/2) \text{ and } W(x) = -\beta K x^2/2$$

- **Future work:** singular case!

Figure: **Welcome to Wuhan University!**

Thanks for your kind attention!